HW3 Solutions

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1 Problem1

a). First note that divergence of a curl is zero. Thus when we integrate $\vec{f}$ over any closed surface, we get

$$\oint f_\perp dA = \oint \vec{f} \cdot d\vec{A} = \int \nabla \cdot \vec{f} dV = 0$$

Thus I choose the surface to be a box with one side (S_{left}) just to the left of the plane and the other side (S_{right}) just to the right of the plane (i.e. it's a Gaussian Pillbox of infinitesimal height). Then

$$\int_{S_{right}} f_{r\perp} dA = \int_{S_{left}} fl \perp dA$$

Since this is true for any box, $f_{r\perp} = f_{l\perp}$ (i.e. $f_\perp$ has the same value on both sides of the plane) and therefore $f_\perp$ is continuous across the plane. b). Similarly notice that

$$\int \vec{f} \cdot d\vec{A} = \int \nabla \vec{a} \cdot d\vec{A} = \oint \vec{a} \cdot d\vec{l} = \oint a_{||} dl,$$

which holds for any surface and corresponding boundary. Thus we can choose the surface to be half of the box mentioned above with the boundary on the plane. The boundary can then be deformed (in any place) such that it is infinitesimally to the left or to the right of the plane. Since the deformations are infinitesimal, $\int \vec{f} \cdot d\vec{A}$ (and therefore $\oint a_{||} dl$) is unchanged as long as $\vec{f}$ is finite. Thus $a_{||}$ is continuous on the surface.

2 Problem2

a). Cherenkov light is produced as long as $\beta \leq \beta_n$, where $\beta_n = 1/n$ is the speed of light in the material. Thus, $\frac{2}{3} < \beta \leq 1$. b). The light doesn't exit the slab as long as the exit angle is $\geq \frac{\pi}{2}$. Using Snell's Law this becomes

$$n \sin \psi \geq n_2 \sin \frac{\pi}{2} = 1,$$

where $n_2 = 1$ is the index of refraction of air. Thus $\frac{2}{3} \leq \beta \leq \frac{1}{n \cos \psi} = \frac{2}{\sqrt{5}}$

3 Problem3

a). After going through a space of length D the slope of the ray should remain unchanged and it's height should be increased by Dx'. That is precisely what happends when you multiply the column vector by M. b). Similarly, after going through a thin lenz the height of the ray should stay the same; and it's slope should be changed such that if it's coming in from infinity (i.e. $x' = 0$) it converges towards the focal point and if it comes through the focal point (i.e. $x' = x/f$) it comes out parallel to the z axis. c). The effect of the converging lenz, a drift space and a diverging lenz will be the product of three matrices

$$M_b(-f)M_a(\delta)M_b(f) = \begin{matrix} 1-\delta/f & \delta \\ -\delta/f^2 & 1+\delta/f \end{matrix} \tag{1}$$

where $M_b(f)$ and $M_a(D)$ are the matrices given in parts b) and a). We can neglect small quatities when compared to 1. Thus the resultant matrix becomes

$$\begin{matrix} 1 & \delta \\ -\delta/f^2 & 1 \end{matrix}, \tag{2}$$

where the δ corresponds to roughly the length of the lenz (and can be neglected for rays where $x'\delta << x$). The new focal length is $\frac{f^2}{\delta} >> f$.

4 Problem4

a).

$$\left(\begin{matrix} \cos\psi \\ \sin\psi e^{i\alpha} \end{matrix} \right) = \left(\begin{matrix} \cos\psi \\ \sin\psi\cos\alpha + i\sin\psi\sin\alpha \end{matrix} \right) = \left(\begin{matrix} \sin\psi\sin\alpha \\ i\sin\psi\sin\alpha \end{matrix} \right) + \left(\begin{matrix} \cos\psi - \sin\psi\sin\alpha \\ 0 \end{matrix} \right) \tag{3}$$

b). No. The above equation can just as easily be written as

$$\left(\begin{matrix} -\sin\psi\sin\alpha \\ i\sin\psi\sin\alpha \end{matrix} \right) + \left(\begin{matrix} \cos\psi + \sin\psi\sin\alpha \\ 0 \end{matrix} \right) \tag{4}$$

which has a circularly polarized piece, whose polarization is opposite to the one in part a).

5 Problem5

a). A polarizer parallel to the x axis is described by a matrix

$$M = \left(\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right) \tag{5}$$

Then a polarizer rotated by θ with respect to the x axis is described by a matrix $\Lambda(\theta)M\Lambda^{-1}(\theta)$, where $\Lambda(\theta)$ is the rotation matrix. Then the sequence of polarizers can be described as

$$\prod_{m=1}^{n} \Lambda^m M \Lambda^{-m} = \Lambda^n (M\Lambda^{-1})^n,$$

where Λ rotates by $\pi/(2n)$ (and consequently Λ^m rotates by $m\pi/(2n)$).

$$\Lambda^{-1} = \left(\begin{array}{cc} \cos(\pi/(2n)) & -\sin(\pi/(2n)) \\ \sin(\pi/(2n)) & \cos(\pi/(2n)) \end{array} \right) = \left(\begin{array}{cc} 1 & -\pi/(2n) \\ \pi/(2n) & 1 \end{array} \right), \tag{6}$$

where the small angle approximation was used. Thus

$$M\Lambda^{-1} = \left(\begin{array}{cc} 1 & \pi/(2n) \\ 0 & 0 \end{array} \right) \tag{7}$$

and you can verify that $(M\Lambda^{-1})^2 = M\Lambda^{-1}$. Thus $(M\Lambda^{-1})^n = M\Lambda^{-1}$, which is simply M in the limit that n goes to infinity. Since M acting on x polarized light just gives back x polarized light and Λ^n rotates it by $\pi/2$, the system turns x polarized light into y polarized light. b). The half wave plate will not affect the component of light perpendicular to it, but will switch the direction of the light parallel to it. From geometry we see that light will go from having an angle of $\pi/4$ with respect to the slow axis to having an angle of $3\pi/4$ with respect to it. Thus it will be parallel to the y axis. c). While both apparati have the same effect on x polarized light, the polarizers will complitely stop y polarized light, and the wave plate will turn it into x polarized light.

6 Problem6

a). If we take a wave plate with a slow axis parallel to the y axis, then it will turn right(left) polarized light into the light having an angle of $\pi/4$ $(-\pi/4)$ with respect to the x axis. We then use a polarizer (at an angle of $\pi/4$) to block out what was left polarized light (which automatically leaves right polarized light untouched). Then place another quarter wave plate (with the slow axis parallel to the x axis) to make the light circularly polarized again. b). Imagine filming the Electric field vector as the light goes through the system and then running the tape backwards. When seen in reverse, right hand polarization is left hand polarization, thus the system blocks right hand polarized light and transmits the left hand polarized light.

7 Problem7

a). For a polarizer parallel to the x axis

$$M_x = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \tag{8}$$

Thus $detM_x = 1 * 0 - 0 * 0 = 0$. The polarizer rotated by angle θ is then described by the matrix $\Lambda(\theta)M_x\Lambda^{-1}(\theta)$. Then

$$detM = det(\Lambda(\theta)M_x\Lambda^{-1}(\theta)) = det\Lambda(\theta) * detM_x * det\Lambda^{-1}(\theta) = 0$$

b). Similarly

$$det(M^\dagger M) = detM^\dagger * detM = 0.$$

Since $detI = 1$, M can not be unitary.

8 Problem8

A wave plate having arbitrary thickness and with one of it's axes oriented parallel to the x axis is described by

$$M_x = \left(\begin{array}{cc} 1 & 0 \\ 0 & e^{i\phi} \end{array} \right). \tag{9}$$

Similarly to problem 7, a rotated wave plate is described by $M = \Lambda(\theta)M_x\Lambda^{-1}(\theta)$. Notice that $\Lambda^{-1} = \Lambda^T = \Lambda^\dagger$.

a). First we show that M_x is unitary.

$$M_x^\dagger M_x = \left(\begin{array}{cc} 1 & 0 \\ 0 & e^{-i\phi} \end{array} \right) * \left(\begin{array}{cc} 1 & 0 \\ 0 & e^{i\phi} \end{array} \right) = I \tag{10}$$

Then

$$M^\dagger M = \Lambda M_x^\dagger \Lambda^{-1}\Lambda M_x \Lambda^{-1} = \Lambda M_x^\dagger M_x \Lambda^{-1} = \Lambda\Lambda^{-1} = I.$$

b).

$$detM = det\Lambda(\theta) * detM_x * det\Lambda^{-1}(\theta) = detM_x = e^{i\phi}$$

Thus $|detM| = |e^{i\phi}| = (\cos^2\theta + \sin^2\theta)^{1/2} = 1$.